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APPLICATION OF ADOMIAN'S DECOMPOSITION PROCEDURE TO THE ANALYSIS OF A BEAM ON RANDOM WINKLER SUPPORT

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Abstract—A stochastic version of the classical problem of an infinite "beam on elastic support" is analyzed using a functional expansion due to Adomian. In this procedure the solution is represented as an infinite series of multiple integrals. It is shown that the solution series converges in the mean square sense provided that the coefficient of variation C_k of the stochastic modulus of subgrade reaction is less than $1/\sqrt{2}$. A fourth-order approximation to the average solution is presented, and it is shown that this solution is valid up to $C_k = 0.5$. A characteristic feature of this solution is that it converges to the solution of the associated deterministic problem if either $C_k \to 0$ or $\Theta \to 0$, where Θ is the ratio of the autocorrelation distance of the random modulus of subgrade reaction to the (deterministic) chartacteristic length of the beam support system. Error estimates are presented for various order approximations of the first and second moments of the solution.

STATEMENT OF THE PROBLEM

The classical model of a beam on elastic (Winkler-type) support is quite useful in a variety of geotechnical problems (e.g. the response of a pile to horizontal loading, the behavior of railways and pipe lines, etc.). This model is governed by the well-known differential equation (Hetenyi, 1964):

$$EI\frac{d^4y(x)}{dx^4} + bk(x)y(x) = bs(x) \tag{1}$$

where EI is the flexural rigidity of the beam, $y(x)$ is the deflection at the point x, b is the width of the beam, $k(x)$ is the value of the modulus of subgrade reaction at x, and $s(x)$ represents the density of external loads along the beam.

Due to the natural variability of most natural soils and the inherent limitation in the density of field testing, there exists a significant uncertainty with respect to the true form of the function $k(x)$. This uncertainty can be incorporated into the analysis by considering $k(x)$ to be a random function of the space coordinate along the beam. In the following, every random quantity will be identified by its dependence on a "realization parameter" ω (a point in the space of "events"). Hence the stochastic equivalent of eqn (1) is written as:

$$EI\frac{d^4y(\omega,x)}{dx^4} + bk(\omega,x)y(\omega,x) = bs(x). \tag{2}$$

A particular case of eqn (2) was studied briefly by Bolotin (1969). A more extensive investigation was presented by Krizek and Alonso (1974). Recently, Baker *et al.* (1989a,b) solved this equation using the assumption that the product of the fluctuation components of $y(\omega,x)$ and $k(\omega,x)$ can be neglected (small fluctuations approximation). The results reported by all the above investigators are quite similar, corresponding to a first-order perturbation of the deterministic solution. The analysis in Baker *et al.* (1989b) showed that

this type of solution is valid if the coefficient of variation C_k of the random function $k(\omega, x)$ is less than approximately 0.1.

Compilation of statistical soil data (Lee *et al.*, 1983) shows that the coefficient of variation of many soil deformation parameters is in the range 0.3–0.35, but may be as high as 0.5. Evidently therefore, the small fluctuation approximation does not provide a solution which is valid in the entire range of practical interest.

In the present work we apply Adomian's decomposition procedure (Adomian, 1963, 1970, 1983) to the analysis of the random differential equation (2). This procedure is in the spirit of the Volterra–Wiener functional expansion, which is finding an increasing use as a tool for the solution of stochastic differential equations (e.g. Beran, 1968; Hori, 1973; Markov, 1987). It will be shown that using this procedure it is possible (at least in principle) to extend the analysis up to $C_k < 1/\sqrt{2}$, and we will present solutions which are valid up to $C_k \leqslant 0.5$.

The procedure is applied to the following specific case.

(a) The external load $s(x)$ consists of a single concentrated force P acting at the point $x = 0$, i.e. $s(x) = P\delta(x)$, where $\delta(x)$ is the Dirac delta "function".

(b) The beam is infinite; and eqn. (2) is solved with the following homogeneous boundary conditions at infinity:

$$y(\omega, x \to \infty) = 0$$
$$y(\omega, x \to -\infty) = 0$$
$$y'(\omega, x \to \infty) = 0$$
$$y'(\omega, x \to -\infty) = 0$$

where $y' = \mathrm{d}y/\mathrm{d}x$.

(c) The function $k(\omega, x)$ is assumed to be a homogeneous low-pass normal function, i.e. $k(\omega, x)$ is jointly normal, with constant trend and standard deviation (k_0, σ_k), and the following spectral density function:

$$\phi_k(l) = \begin{cases} R_k\sigma_k^2/\pi & \text{for } |l| \leqslant \pi/(2R_k) \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

where $\phi_k(l)$ and R_k are the spectral density function and autocorrelation distance of $k(\omega, x)$, respectively, and l is the "wave number" (spatial frequency).

The motivation for this characterization of $k(\omega, x)$ is that for given values of k_0, σ_k and R_k, this model represents the least prejudiced assignments of probabilities in the Information Theory sense (Baker and Zeitoun, 1987). Recalling that the stochastic characterization of $k(\omega, x)$ was motivated by uncertainty due to lack of information (resulting from limitation in the density of field testing), this type of model appears reasonable.

THE DETERMINISTIC SOLUTION

The general solution of an infinite deterministic beam on Winkler support [eqn. (1)] is well known. This solution can be written in the form (Salvadurai, 1970):

$$y_d(X) = \frac{1}{2k_0\lambda} \int_{U=-\infty}^{\infty} s(U) g_\lambda(X-U)\, \mathrm{d}U \tag{4}$$

where:

$$\lambda = \sqrt[4]{\frac{4EI}{k_0 b}} \tag{5}$$

$$g_\lambda(u/\lambda) = [\cos(|u/\lambda|) + \sin(|u/\lambda|)]\, e^{-|u/\lambda|} \tag{6}$$

and $X = x/\lambda$; $U = u/\lambda$; $s(U) = \lambda s(x)$. λ is the "characteristic length" of the beam–support system, X and U are non-dimensional coordinates along the beam, and $s(U)$ is the density of external loading per unit length of the non-dimensional coordinate U.

Notice that the term $g_\lambda(u)/(2k_0\lambda b)$ is the Green function of the deterministic linear differential operator $\mathbf{L}\{\cdot\}$:

$$\mathbf{L}\{\cdot\} = EI\frac{d^4\{\cdot\}}{dx^4} + bk_0\{\cdot\}. \tag{7}$$

The Green function $g_\lambda(u)$ has the form of a strongly damped cos ( ) wave, with maximum value of unity at $u = 0$.

In the present problem $s(U) = P\delta(U)$ so that eqn (4) yields:

$$y_d(X) = y_d(0) g_\lambda(X) \tag{8}$$

$$y_d(0) = P/(2k_0\lambda). \tag{9}$$

We will refer to $y_d(X)$ as the "deterministic solution".

SMALL FLUCTUATION APPROXIMATION—SUMMARY OF RESULTS

For later reference we summarize here the main results derived by Baker *et al.* (1989a,b) on the basis of the small fluctuation approximation.

- The average value of $y(\omega, x)$ equals the deterministic solution i.e.:

$$\bar{y}(x) = \langle y(\omega, x)\rangle = y_d(x) \tag{10}$$

  where $\langle\cdot\rangle$ stands for the expected value operator, i.e. ensemble average with respect to the realizations.
- The fluctuation component of $y(\omega, x)$ is given by:

$$\hat{y}(\omega, X) = \frac{-1}{2k_0\lambda}\int_{-\infty}^{\infty} g_\lambda(X-U) y_d(U)\hat{k}(\omega, U)\, dU \tag{11}$$

  where $\hat{k}(\omega, x)$ and $\hat{y}(\omega, x)$ are the fluctuation components of $k(\omega, x)$ and $y(\omega, x)$ respectively, i.e.:

$$\hat{y}(\omega, x) = y(\omega, x) - \bar{y}(x)$$
$$\hat{k}(\omega, x) = k(\omega, x) - k_0.$$

- Equations (10), (11) are valid with an error not exceeding 5% provided that $C_k = \sigma_k/k_0 \leqslant 0.1$.
- The basic premise of this approximation is that for every realization ω it is possible to neglect the term $\hat{y}(\omega, x)\hat{k}(\omega, x)$ in comparison with $[y_d(x)k_0 + y_d(x)\hat{k}(\omega, x) + k_0\hat{y}(\omega, x)]$.

ADOMIAN'S DECOMPOSITION PROCEDURE

Decomposing $k(\omega, x)$ into deterministic and stochastic components, and using the definition of the operator $\mathbf{L}\{\cdot\}$ [eqn. (7)] makes it possible to write the stochastic differential eqn (2) in an operator form as:

$$\mathbf{L}\{y(\omega,x)\} = b[P\delta(x) - \hat{k}(\omega,x)y(\omega,x)]. \tag{12}$$

Since $\mathbf{L}\{\cdot\}$ is a linear operator, its inverse $\mathbf{L}^{-1}\{\cdot\}$ is also linear, hence operating on eqn (12) with $\mathbf{L}^{-1}\{\cdot\}$ gives:

$$y(\omega,x) = bP\mathbf{L}^{-1}\{\delta(x)\} - b\mathbf{L}^{-1}\{\hat{k}(\omega,x)y(\omega,x)\}. \tag{13}$$

The inverse operator $\mathbf{L}^{-1}\{\cdot\}$ can be written as:

$$\mathbf{L}^{-1}\{\cdot\} = \frac{1}{2k_0\lambda b}\int_{u=-\infty}^{\infty} g_\lambda(x-u)\{\cdot\}\,du.$$

The term $bP\mathbf{L}^{-1}\{\delta(x)\}$ is just $y_d(x)$ and hence eqn. (13) becomes:

$$y(\omega,x) = y_d(x) - \frac{1}{2k_0\lambda}\int_{u=-\infty}^{\infty} g_\lambda(x-u)\hat{k}(\omega,u)y(\omega,u)\,du. \tag{14}$$

Equation (14) is a stochastic Volterra equation for the function $y(\omega, x)$. Adomian (1963) suggested a procedure for the solution of these types of equations. His procedure consists of the following two steps.

- Assume that the unknown function $y(\omega, x)$ can be represented as a sum of undefined functions $y_i(\omega, x)$ as follows:

$$y(\omega,x) = y_d(x) + \sum_{i=1}^{\infty} y_i(\omega,x). \tag{15}$$

Substituting (15) into (14) gives:

$$y_d(x)+y_1(\omega,x)+y_2(\omega,x)+\cdots+y_n(\omega,x) = y_d(x) - \frac{1}{2k_0\lambda}\int_{u=-\infty}^{\infty} g_\lambda(x-u)\hat{k}(\omega,u) \times [y_d(u)+y_1(\omega,u)+y_2(\omega,u)+\cdots+y_n(\omega,u)]\,du. \tag{16}$$

- Since the functions $y_i(\omega, x)$, $i = 1, 2, \ldots \infty$, are still not specified, it is possible to make the following identification:

$$\begin{aligned}
y_1(\omega,x) &= \frac{-1}{2k_0\lambda}\int_{u=-\infty}^{\infty} g_\lambda(x-u)\hat{k}(\omega,u)y_d(u)\,du \\
y_2(\omega,x) &= \frac{-1}{2k_0\lambda}\int_{u=-\infty}^{\infty} g_\lambda(x-u)\hat{k}(\omega,u)y_1(w,u)\,du \\
y_3(\omega,x) &= \frac{-1}{2k_0\lambda}\int_{u=-\infty}^{\infty} g_\lambda(x-u)\hat{k}(\omega,u)y_2(\omega,u)\,du \\
&\vdots \qquad\qquad\qquad \vdots \\
y_n(\omega,x) &= \frac{-1}{2k_0\lambda}\int_{u=-\infty}^{\infty} g_\lambda(x-u)\hat{k}(\omega,u)y_{n-1}(\omega,u)\,du.
\end{aligned} \tag{17}$$

In order for this system of integral equations to be formally equivalent to eqn (16), it is necessary to satisfy the following "consistency relation":

$$\lim_{n\to\infty}\left[\int_{u=-\infty}^{\infty} g_\lambda(x-u)\hat{k}(\omega,u)y_n(\omega,u)\,\mathrm{d}u\right]=0.$$

However, if series (15) converges (in some sense) then $\lim_{n\to\infty} y_n(\omega,x)\to 0$ and the consistency relation is satisfied.

The system of integral eqns (17) possesses a number of very attractive properties, as follows.

- Each element $y_i(\omega,x)$ of this system is given in terms of lower-order terms only; hence the procedure does not require a "closure approximation" as is the case in the "hierarchy"-type methods (Soong, 1973).
- The derivation of eqns (17) requires no assumptions about the nature of the random function $\hat{k}(\omega,x)$. Nevertheless, we will show that in order to ensure the convergence of series (15) it is necessary to restrict the class of admissible functions $\hat{k}(\omega,x)$.
- It is important to emphasize that Adomian's decomposition is not a perturbation technique, and no small parameter is involved (Adomian, 1983).

Back substitution of the terms $y_i(\omega,x)$ in eqns (17) yields an explicit representation of these terms in the form of multiple integrals as follows:

$$y_1(\omega,x)=\frac{-y_d(0)}{2k_0\lambda}\int_{u_1=-\infty}^{\infty} g_\lambda(x-u_1)\hat{k}(\omega,u_1)g_\lambda(u_1)\,\mathrm{d}u_1$$

$$y_2(\omega,x)=\frac{-y_d(0)}{(2k_0\lambda)^2}\int_{u_1=-\infty}^{\infty}\int_{u_2=-\infty}^{\infty}[g_\lambda(x-u_1)g_\lambda(u_1-u_2)][\hat{k}(\omega,u_1)\hat{k}(\omega,u_2)]g_\lambda(u_2)\,\mathrm{d}u_2\,\mathrm{d}u_1$$

$$\vdots \qquad\qquad \vdots$$

$$y_n(\omega,x)=\frac{(-1)^n y_d(0)}{(2k_0\lambda)^n}\int_{u_1=-\infty}^{\infty}\cdots\int_{u_n=-\infty}^{\infty}[g_\lambda(x-u_1)\cdots g_\lambda(u_{n-1}-u_n)]$$
$$\times[\hat{k}(\omega,u_1)\cdots\hat{k}(\omega,u_n)]g_\lambda(u_n)[\mathrm{d}u_n\cdots\mathrm{d}u_1]. \quad (18)$$

RANGE OF VALIDITY OF THE SOLUTION

The series representation of the solution [eqn (15)] is meaningful only if it converges. In order to study the convergences properties of this series, define the nth-order approximation of the solution as:

$$y^{(n)}(\omega,x)=y_d(x)+\sum_{i=1}^{n} y_i(\omega,x).$$

The absolute error associated with this approximation is:

$$e^{(n)}(\omega)=\|y(\omega,x)-y^{(n)}(\omega,x)\|=\left\|\sum_{i=n+1}^{\infty} y_i(\omega,x)\right\|$$

where $\|\cdot\|$ is the upper bound operator, i.e. maximum over x of the absolute value of the operand.

The solution series converges if the absolute error vanishes as $n\to\infty$. Realizing however that $e^{(n)}(\omega)$ is a random variable, it is necessary to specify the sense of such a limiting

operation. In the present work we consider "mean square convergence" only. By definition a sequence of random variables $U_n(\omega)$ converges in the mean square to the constant U if:

$$\lim_{n\to 0} \langle [U_n(\omega) - U]^2 \rangle = 0.$$

The condition for mean square convergence of the solution series is therefore:

$$\lim_{n\to\infty} \langle [e^{(n)}(\omega) - 0]^2 \rangle = \lim_{n\to 0} \langle [e^{(n)}(\omega)]^2 \rangle = 0. \tag{19}$$

It can be shown (Parzen, 1960) that mean square convergence is a stronger requirement than "convergence in probability" in the sense that the former implies the latter.

Equation (19) can be written as:

$$\lim_{n\to\infty} A^{(n)} = 0 \tag{20}$$

where

$$A^{(n)} = \langle [e^{(n)}(\omega)]^2 \rangle = \left\| \sum_{i=n+1}^{\infty} \sum_{j=n-1}^{\infty} \langle y_i(\omega, x) y_j(\omega, x) \rangle \right\|. \tag{21}$$

In Appendix A it is shown that $A^{(n)}$ is bounded by the expression:

$$A^{(n)} \leqslant y_d^2(0) \left[\frac{Z}{1-Z}\right]^2 \left\{2n + \frac{1+Z}{1-Z}\right\} Z^{2n} \tag{22}$$

where $Z = \sqrt{2}C_k$.

This relation shows clearly that the limit $\lim_{n\to\infty} A^{(n)} = 0$ is satisfied provided that $Z < 1$, or equivalently:

$$C_k < 1/\sqrt{2}. \tag{23}$$

It should be realized that eqn (22) represents an upper bound on the errors. In other words, eqn (23) is a sufficient but probably not necessary condition for the convergence of the solution series. In particular, the derivation of the bound for $A^{(n)}$ (Appendix A) is based on the relation $\| \rho_k(x) \| \leqslant 1$, [where $\rho_k(x)$ is the autocorrelation function of $k(\omega, x)$]. Therefore the bound on $A^{(n)}$ corresponds to the random variable limit of the function $k(\omega, x)$. It may be possible to obtain a tidier limit on the errors (and hence a broader convergence criterion) by considering a more realistic correlation structure in the error analyses. An advantage of the present approach is, however, the fact that the results are valid for all possible autocorrelation functions $\rho_k(x)$. As a result of these considerations we cannot associate any physical significance with the limiting value $C_k = 1/\sqrt{2}$. This value is merely the consequence of the particular upper bounds used in Appendix A.

Equation (23) was derived as the requirement for mean square convergence of the solution series (15). Using a similar technique it is possible to show that:

$$e_m^{(n)} = \sum_{l=1}^{m+1} b_{m,l} \alpha_{m,l}^{(n)} \tag{24}$$

$$\alpha_{m,l}^{(n)} \leqslant \frac{Z^{l(n+1)}}{(1-Z)^m} \left[l\left(n + \frac{1}{1-z}\right) + (m-l)\frac{Z}{1-Z} - 1 \right] \tag{25}$$

where $e_m^{(n)}$ is the error resulting from taking only n terms in the series representation of the mth-order moment of $y(\omega, x)$ about zero, and $b_{m,l}$ are the binomial coefficients.

Inspection of eqn (25) shows that if $Z < 1$ then each one of the terms $\alpha_{m,l}^{(n)} \to 0$ when $n \to \infty$. Hence for any finite order moment m, and $Z < 1$, the error $e_m^{(n)} \to 0$ when $n \to \infty$, i.e. moments of all order converge if $C_k < 1/\sqrt{2}$.

The convergence criteria obtained for the present particular problem are stronger than other convergence critera for stochastic differential equations. For example:

- It can be shown (Beran, 1968; Hori, 1973) that in the framework of perturbation theory, a sufficient condition for the convergence of the perturbation series can be written as:

$$\frac{\| \langle [\hat{k}(\omega, x)]^m \rangle \|}{[k_0]^m} \ll 1 \quad \text{for all } m = 1, \ldots \infty. \tag{26}$$

  For $m = 1$ this requirement is satisfied trivially by virtue of the definition of the fluctuation component. For $m = 2$ eqn (26) is equivalent to $C_k^2 \ll 1$ (compared with our result $C_k < 1/\sqrt{2}$). For higher values of m the physical significance of eqn (26) is not very clear.
- Markov (1987) showed that a sufficient condition for convergence of the Volterra–Wiener expansion is again eqn (26), but in addition it is required that:

$$\frac{\| \langle [\hat{k}(\omega, x)]^{m+1} \rangle \|}{[k_0]^{m+1}} = \mathcal{O}\left\{ \frac{\| \langle [\hat{k}(\omega, x)]^m \rangle \|}{[k_0]^m} \right\} \tag{27}$$

  where $\mathcal{O}\{\cdot\}$ is the "order of magnitude" operator.
- Adomian (1983) proved convergence of the present decomposition method for the case when both $k(\omega, x)$ and x are bounded. In the present notation his convergence criteria may be written as:

$$\| k(\omega, x) \| \leqslant k_{\max} \quad \text{(in every realization } \omega) \tag{28}$$

  and

$$\| x \| \leqslant x_{\max}. \tag{29}$$

  Evidently eqn (28) is not consistent with the assumption that $k(\omega, x)$ is a normal function. Nevertheless, we have attempted to apply Adomian's type of analysis to our problem using condition (28) but not (29). In that case it can be shown that instead of a restriction on the magnitude of C_k one gets the requirement:

$$\frac{k_{\max}}{k_0} < 1 + 1/\sqrt{2}.$$

  It should be realized that this equation represents a very severe restriction. For example, it would not be satisfied for a uniform distribution on the range 0–$k_{\max}$. Nor would it be satisfied for any distribution in which the average is located in the center of the interval.

We may conclude that the constraint $C_k < 1/\sqrt{2}$ is fairly mild compared with other available criteria for convergence of series-type solutions of stochastic differential equations. It is realized that we have obtained this result for a particular problem only, and it is the consequence of the fact that our Green function g_λ is very "concentrated" near the origin, so that the bound $\| g_\lambda(u) \| \leqslant \sqrt{2}\, e^{-|u/\lambda|}$ [eqn (A11)] applies.

THE AVERAGE SOLUTION

General

Using eqn (15), the average solution $\bar{y}(x) = \langle y(\omega, x)\rangle$ admits the following representation:

$$\bar{y}(x) = y_d(x) + \sum_{i=1}^{\infty} \bar{y}_i(x)$$

$$\bar{y}_i(x) = \langle y_i(\omega, x)\rangle \quad i = 1, \ldots, \infty.$$

Utilizing the form of the general term $y_i(\omega, x)$ in eqn (18), and the correlation structure of normal random functions [eqn. (A3)], yields:

$$\bar{y}(x) = y_d(x) + \bar{y}_2(x) + \bar{y}_4(x) + \cdots \bar{y}_{2i}(x) = y_d(x) + \sum_{i=1}^{\infty} \bar{y}_{2i}(x) \tag{30}$$

with:

$$\bar{y}_{2i}(x) = \frac{(C_k)^{2i} y_d(0)}{(2\lambda)^{2i}} \int_{u_1=-\infty}^{\infty} \cdots \int_{u_{2i}=-\infty}^{\infty} [g_\lambda(x-u_1)\cdots g_\lambda(u_{2i-1}-u_{2i})] \times \left[\sum_{j=1}^{(2i-1)} a_{2i,j}(u_1, \ldots, u_{2i})\right] g_\lambda(u_{2i}) \times (\mathrm{d}u_{2i}, \ldots, \mathrm{d}u_1) \tag{31}$$

where the parameters $a_{i,j}$ are defined in Appendix A.

Comparing eqn (30) with the small fluctuation result $\bar{y}(x) = y_d(x)$ [eqn (13)], it is clear that with respect to the average solution the small fluctuation approximation corresponds to a "zero-order" Adomian's expansion.

Evidently high-order terms in the expansion for $\bar{y}(x)$, become increasingly complex and are quite difficult to evaluate even numerically. Therefore the real test of the usefulness of this procedure is whether or not low-order terms of the expansion provide significant improvement over the small fluctuation solution. In order to allow such a comparison we proceed to evaluate the errors associated with the various order Adomian's expansions for the solution of the average displacement.

Errors of the average solution

The errors associated with various order approximations of the average solution series can be obtained by setting $m = 1$ in eqn (24). Noticing, however, that eqn (30) contains only even terms, a better error estimate can be obtained for this case. Utilizing the same arguments as in Appendix A and taking into account the structure of eqn (30) results in the following error bound:

$$\varepsilon_{\bar{y}}^{(2n)} = \frac{\|\bar{y}(x) - \bar{y}^{(2n)}(x)\|}{y_d(0)} \leqslant \frac{Z^{2(n+1)}}{1-Z^2}\left[2n + \frac{1+Z^2}{1-Z^2}\right] \tag{32}$$

where $\varepsilon_{\bar{y}}^{(2n)}$ represents the relative error in $\bar{y}(x)$ resulting from truncating series (30) after n terms, and as before $Z = \sqrt{2}C_k$.

Equation (32) shows that series (30) converges if $C_k < 1/\sqrt{2}$. In Fig. 1 we show the upper bound of $\varepsilon_{\bar{y}}^{(2n)}$ as a function of C_k for different n values, with $n = 0$ corresponding to the deterministic solution. This upper bound is calculated using eqn. (32).

Inspection of Fig. 1 shows that in order to use the zero-order solution (small fluctuation approximation) with less then 5% error, it is necessary to restrict the coefficient of variation to $C_k \leqslant 0.15$. This is somewhat higher than the range $C_k \leqslant 0.1$ reported by Baker *et al.* (1988b). It must be realized, however, that the two ranges refer to different definitions of the error.

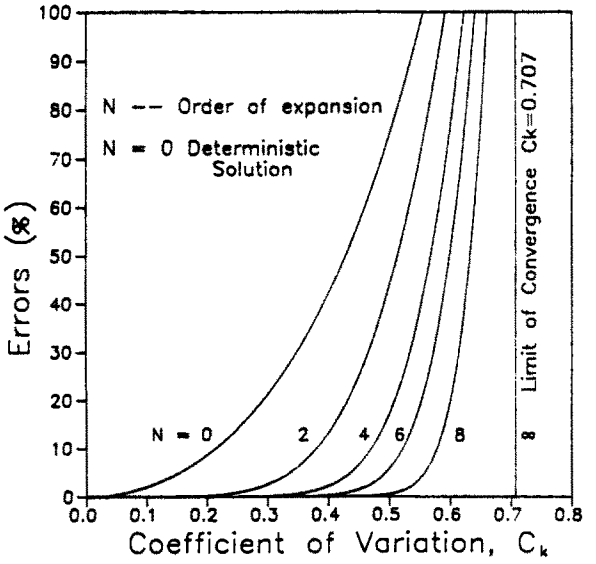

Fig. 1. Upper bound on the errors of the average solution.

Keeping the accuracy requirement at the 5% level and including only the first non-zero term of the expansion for $\bar{y}(x)$; $\{\bar{y}_2(x)\}$, increases the range of validity of the analysis to $C_k \leqslant 0.25$–0.30. In order to extend the analysis up to $C_k = 0.5$, it is necessary to include at least $\bar{y}_4(x)$ and possibly also $\bar{y}_6(x)$. In the following we present explicit results for $\bar{y}_2(x)$ and $\bar{y}_4(x)$.

The second-order term

Specifying eqn (31) for $i = 1$ gives:

$$\bar{y}_2(x) = \frac{y_d(0)C_k^2}{(2\lambda)^2} \int_{u_1=-\infty}^{\infty} \int_{u_2=-\infty}^{\infty} g_\lambda(x-u_1) g_\lambda(u_1-u_2) \rho_k(u_1-u_2) g_\lambda(u_2)\, \mathrm{d}u_2\, du_1. \quad (33)$$

Noticing that eqn (33) has the form of a double convolution integral, and recalling that we have a particularly simple representation of the correlation structure in terms of the spectrum [eqn (3)], it appears that the application of Fourier Transform technique will simplify the analysis. In the following we use the convention that capital letters signify a transformed quantity. Taking the Fourier Transform of eqn (33) yields:

$$\bar{Y}_2(l) = \frac{y_d(0)C_k^2}{(2\lambda)^2} G_\lambda^2(l) T(l) \quad (34)$$

where:

$$\bar{Y}_2(l) = \mathbf{F}_T\{\bar{y}_2(x)\}$$
$$G_\lambda(l) = \mathbf{F}_T\{g_\lambda(x)\}$$
$$T(l) = \mathbf{F}_T\{g_\lambda(x)\rho_k(x)\}$$

and $\mathbf{F}_T\{\cdot\}$ stands for the Fourier Transform operator.

Utilizing the convolution theorem in the frequency domain, Papoulis (1962) gives:

$$T(l) = \frac{1}{2\pi\sigma_k^2} \int_{l_1=-\infty}^{\infty} G_\lambda(l-l_1)\phi_k(l_1)\, \mathrm{d}l_1 = \frac{R_k}{\pi} \int_{l_1=-\pi/2R_k}^{\pi/2R_k} G_\lambda(l-l_1)\, \mathrm{d}l_1 \quad (35)$$

where the form of the spectrum specified in eqn (3) was used.

Substituting eqn (35) into eqn (34) and using (9) gives:

$$\bar{Y}_2(l) = \frac{y_d(0)R_k C_k^2}{\pi(2\lambda)^2} G_\lambda^2(l) \int_{l_1=-\pi/2R_k}^{\pi/2R_k} G_\lambda(l-l_1)\,\mathrm{d}l_1. \tag{36}$$

In order to evaluate $G_\lambda(l)$, recall that $g_\lambda(x)/(2k_0\lambda b)$ is the Green function of the operator $\mathbf{L}\{\cdot\}$ [eqn (7)]. Consequently $g_\lambda(x)$ satisfies the following differential equation:

$$EI\frac{\mathrm{d}^4 g_\lambda(x)}{\mathrm{d}x^4} + bk_0 g_\lambda(x) = 2bk_0\lambda\delta(x).$$

Taking the Fourier Transform of this equation, and rearranging:

$$G_\lambda(l) = \frac{2bk_0\lambda}{EIl^4 + bk_0}.$$

Using eqn (5) this expression can be written in a non-dimensional form as:

$$G_\lambda(L) = 8\lambda G(L)$$

$$G(L) = \frac{1}{L^4+4}$$

$$L = \lambda l \tag{37}$$

where L is a non-dimensional wave number. Notice that $G(L)$ is a purely algebraic form, independent of all the parameters of the problem.

Substituting (37) into (36) and introducing the non-dimensional quantities $\Theta = R_k/\lambda$ and $L_1 = \lambda l_1$ gives:

$$\bar{Y}_2(L) = \frac{128 y_d(0) C_k^2 \Theta}{\pi} G^2(L) \int_{L_1=-\pi/2\Theta}^{\pi/2\Theta} G(L-L_1)\,\mathrm{d}L_1. \tag{38}$$

The parameter $\Theta = R_k/\lambda$ together with C_k are the two fundamental characteristics of the problem.

The finite integral with respect to L_1 was evaluated numerically using six point Gaussian quadrature, and the result was transformed back into the real domain using a standard numerical Fourier Transform subroutine.

The fourth-order term

Specifing eqn (30) for $i = 2$, using (A3),(A4) and taking the Fourier Transform of the result, gives:

$$\bar{Y}_4(l) = \frac{y_d(0)C_k^4}{(2\lambda)^4}[H_{4,1}(l) + H_{4,2}(l) + H_{4,3}(l)] \tag{39}$$

where

$$H_{4,1}(l) = \mathbf{F}_T\left\{\int_{u_1=-\infty}^{\infty}\cdots\int_{u_4=-\infty}^{\infty} f(x,u_1,\ldots,u_4)[\rho_k(u_1-u_2)\rho_k(u_3-u_4)](\mathrm{d}u_4,\ldots,\mathrm{d}u_1)\right\}$$

$$H_{4,2}(l) = \mathbf{F}_T\left\{\int_{u_1=-\infty}^{\infty}\cdots\int_{u_4=-\infty}^{\infty} f(x,u_1,\ldots,u_4)[\rho_k(u_1-u_3)\rho_k(u_2-u_4)](\mathrm{d}u_4,\ldots,\mathrm{d}u_1)\right\}$$

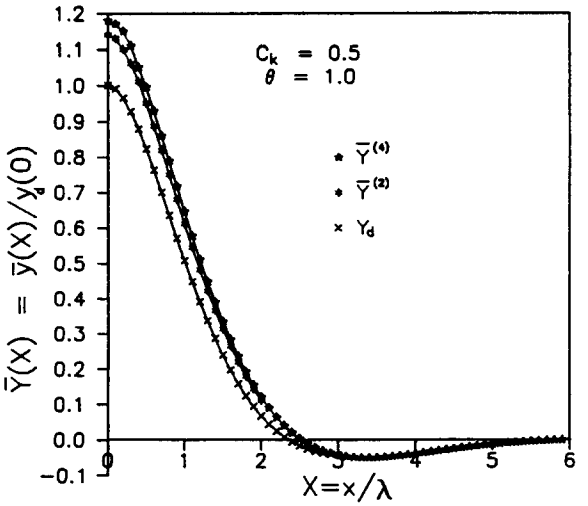

Fig. 2. Relative magnitude of various order approximations for the average solution.

$$H_{4,3}(l) = \mathbf{F}_T\left\{\int_{u_1=-\infty}^{\infty}\cdots\int_{u_4=-\infty}^{\infty} f(x,u_1,\ldots,u_4)[\rho_k(u_1-u_4)\rho_k(u_2-u_3)](\mathrm{d}u_4,\ldots,\mathrm{d}u_1)\right\} \tag{40}$$

and

$$f(x,u_1,\ldots,u_4) = g_\lambda(x-u_1)g_\lambda(u_1-u_2)g_\lambda(u_2-u_3)g_\lambda(u_3-u_4)g_\lambda(u_4). \tag{41}$$

The evaluation of the Fourier Transforms $H_{4,1}(l)$; $H_{4,2}(l)$; $H_{4,3}(l)$ is somewhat lengthy and therefore it is presented in Appendix B. Substituting these functions [(B1), (B2) and (B3)] into (39) gives:

$$\bar{Y}_4(l) = \frac{C_k^4 y_d(0)}{(2\lambda)^4}\left[\frac{R_k}{\pi}\right]^2 G_\lambda^2(l)\int_{l_1=-\pi/2R_k}^{\pi/2R_k}\int_{l_2=-\pi/2R_k}^{\pi/2R_k} G_\lambda(l-l_1)\{G_\lambda(l)G_\lambda(l-l_2) + G_\lambda(l-l_1-l_2)[G_\lambda(l-l_1)+G_\lambda(l-l_2)]\}\,\mathrm{d}l_2\,\mathrm{d}l_1.$$

Transforming this expression to the non-dimensional quantities Θ; L_1; L_2 and using the non-dimensional form of G [eqns (37)] gives:

$$\bar{Y}_4(L) = \frac{y_d(0)}{2}\left[\frac{64C_k^2\Theta}{\pi}\right]^2 G^2(L)\int_{L_1=-\pi/2\Theta}^{\pi/2\Theta}\int_{L_2=-\pi/2\Theta}^{\pi/2\Theta} G(L-L_1)\{G(L)G(L-L_2) + G(L-L_1-L_2)[G(L-L_1)+G(L-L_2)]\}\,\mathrm{d}L_2\,\mathrm{d}L_1. \tag{42}$$

The finite integrals with respect to L_1 and L_2 were evaluated numerically using six point Gaussian quadrature, and the result was transformed back into the real domain using a standard numerical Fourier Transform subroutine.

Results

The second- and fourth-order approximations to the solution are defined as:

$$\bar{y}^{(2)}(x) = y_d(x)+\bar{y}_2(x)$$
$$\bar{y}^{(4)}(x) = y_d(x)+\bar{y}_2(x)+\bar{y}_4(x).$$

Figure 2 shows the relative magnitude of $y_d(x)$, $\bar{y}^{(2)}(x)$ and $\bar{y}^{(4)}(x)$ for $C_k = 0.5$ and

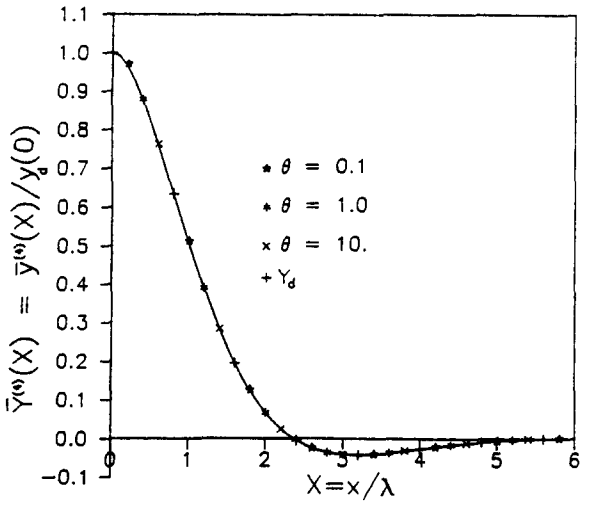

Fig. 3. Fourth-order solution for the displacement—$C_k = 0.1$.

$\Theta = 1$. The function $\bar{y}^{(4)}(x)$ is shown in Figs 3, 4 and 5. All the y values shown in these figures are normalized with respect to $y_d(0)$.

Figure 3 represents the case $C_k = 0.1$ (limit of validity of the small fluctuation solution) and three values of Θ: $\Theta = 0.1$, $\Theta = 1.0$ and $\Theta = 10$. As expected, for such a low value of C_k, $y_d(x)$ provides an excellent approximation to $\bar{y}^{(4)}(x)$ regardless of the Θ value. This observation supports the results of Baker *et al.* (1989a,b).

Figure 4 represents the case $C_k = 0.3$ (limit of validity of the second-order solution), and the same Θ values as Fig. 4. According to Fig. 1 for $C_k = 0.2$ the error of $y_d(x) = \bar{y}_0(x)$ should be of the order of 30%. Figure 4 shows, however, that the maximum difference between $y_d(x)$ and $\bar{y}^{(4)}(x)$ is less then 5%. This observation suggests that the upper bounds of the error estimate shown in Fig. 1 are conservative, and we may probably utilize the results at higher C_k values than those implied by Fig. 1.

Figure 5 shows the results for $C_k = 0.5$. In this case the difference between $y_d(x)$ and $\bar{y}^{(4)}(x)$ starts being significant (20% at the origin), and the effect of Θ noticeable. An important feature of this figure is the fact that for $\Theta \leqslant 0.1$, $\bar{y}^{(4)} \approx y_d(x)$ despite the fact that $C_k = 0.5$. In order to understand this phenomenon, recall that $\Theta = R_k/\lambda$, so that $\Theta \to 0$ implies $R_k \to 0$. In this limit the function $k(\omega, x)$ degenerates into a "white noise". In the white noise limit the fluctuations of $k(\omega, x)$ are so "rapid" that the relatively rigid beam cannot follow them, and as a result these fluctuations are not "effective" in modifying the

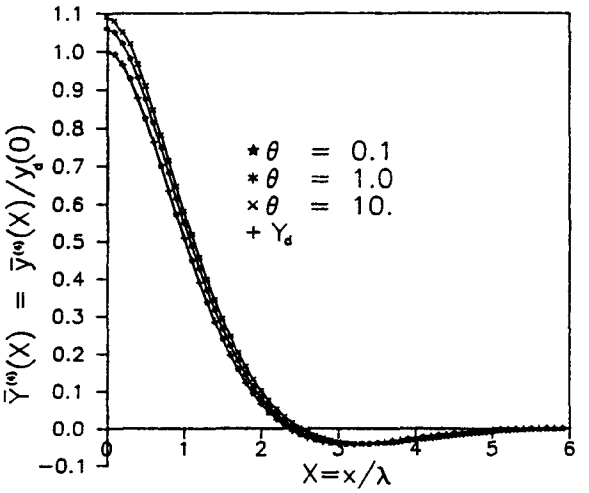

Fig. 4. Fourth-order solution for the displacement—$C_k = 0.3$.

deterministic solution. It is reasonable to expect that in the white noise limit this type of behavior is valid in general; not only for the fourth-order term.

We have calculated $y^{(4)}(x)$ also for $\Theta = 100$ and the results were identical to those corresponding to $\Theta = 10$. This result corresponds to the physical situation where $R_k \gg \lambda$ so that within the range of influence of the concentrated load P, the random function $k(\omega, x)$ is essentially a constant of uncertain magnitude [i.e. $k(\omega, x)$ degenerates into a single random variable]. Further increase in Θ cannot have any effect on the solution. We shall call this limiting case the "random variable limit". It is seen that as far as the average solution is concerned, the range $\Theta = 0.1$–10 covers the complete spectrum of physical situations from the "white noise" limit to the "random variable" limit. Notice that these terms refer here to the beam support system as a whole, not just the random function $k(\omega, x)$. In other words the parameter of significance is not the autocorrelation distance R_k, but the ratio of this distance to the (deterministic) characteristic length of the beam support system λ. It may be verified that all aspects of the solution (not just the average) depend on the two non-dimensional parameters C_k and Θ.

VARIANCE OF THE SOLUTION

General

The variance function (squared standard deviation) of the displacement is defined as:

$$V(x) = \langle [y(\omega, x) - \bar{y}(x)]^2 \rangle = \left\langle \left[\sum_{i=1}^{\infty} y_i(\omega, x) \right]^2 \right\rangle - \left[\sum_{i=1}^{\infty} \bar{y}_i(x) \right]^2. \tag{43}$$

The nth-order approximation to $V(x)$ is:

$$V^{(n)}(x) = \left\langle \left[\sum_{i=1}^{n} y_i(\omega, x) \right]^2 \right\rangle - \left[\sum_{i=1}^{m} \bar{y}_{2i}(x) \right]^2 \tag{44}$$

where $m = n/2$ if n is even, and $m = (n-1)/2$ if n is odd.

It is convenient to write $V^{(n)}(x)$ in the form:

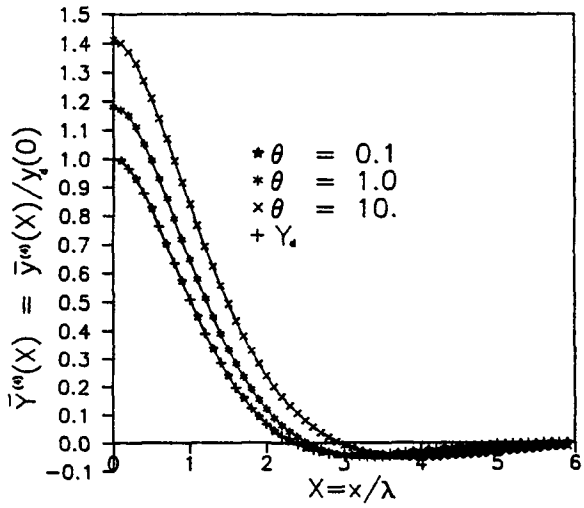

Fig. 5. Fourth-order solution for the displacement—$C_k = 0.5$.

$$V^{(n)}(x) = \sum_{i=1}^{n} V_i(x) \tag{45}$$

where the terms $V_i(x)$ are given by:

$$V_i(x) = V^{(i)}(x) - V^{(i-1)}(x) \tag{46}$$

and $V_1(x) = V^{(1)}(x)$.

Considering the correlation structure of $k(\omega, x)$ and using equations (44) and (46), the first four terms $V_i(x)$, $i = 1, \ldots, 4$, are given by:

$$V_1(x) = \langle y_1^2(\omega, x)\rangle$$
$$V_2(x) = \langle y_2^2(\omega, x)\rangle - \bar{y}_2^2(x)$$
$$V_3(x) = \langle y_3^2(\omega, x)\rangle + 2\langle y_1(\omega, x) y_3(\omega, x)\rangle$$
$$V_4(x) = [\langle y_4^2(\omega, x)\rangle + 2\langle y_2(\omega, x) y_4(\omega, x)\rangle] - [2\bar{y}_2(x)\bar{y}_4(x) + \bar{y}_4^2(x)].$$

Using eqns (18), (A3) and (A4) it is possible to derive explicit expressions for the different terms $V_i(x)$ employing essentially the same procedure that was used for the derivation of the average solution. For the sake of brevity these expressions and numerical results based on them will be reported elsewhere. It is of interest, however, to compare the present expression for $V(x)$ with the solution based on the small fluctuation approximation (Baker *et al.*, 1989a,b) and establish the accuracy of this type of approximation. Recalling that $\bar{y}_1(x) = 0$, we have $\hat{y}_1(\omega, x) = y_1(\omega, x)$. Moreover, comparing the expression for $y_1(\omega, x)$ [first one of eqns (18)] with the small fluctuation solution for $\hat{y}(\omega, x)$ [eqn (11)] one can see that the small fluctuation solution for the fluctuation component of $y(\omega, x)$ is simply $y_1(\omega, x)$. It follows therefore that the solution for the standard deviation presented by Baker *et al.* (1988a,b) corresponds to the first-order term $\{V^{(1)}(x) = V_1(x)\}$ of the present solution.

In the next section we present error estimates for various order approximations of the variance function in order to assess the accuracy of the small fluctuation solution for $V(x)$.

Errors of the variance function

We present in this section a derivation of the errors of the variance function. A similar procedure was used in the derivation of the error of the mth-order moments about zero. The relative error of the nth-order approximation to the variance is defined as:

$$\varepsilon_V^{(n)} = [\| V(x) - V^{(n)}(x) \|]/y_d(0) = [\varepsilon_1^{(n)} + \varepsilon_2^{(n)}]/y_d(0) \tag{47}$$

where

$$\varepsilon_1^{(n)} = \left\| \left\langle \left[\sum_{i=1}^{\infty} y_i(\omega, x) \right]^2 \right\rangle - \left\langle \left[\sum_{i=1}^{n} y_i(\omega, x) \right]^2 \right\rangle \right\|$$

$$\varepsilon_2^{(n)} = \left\| \left[\sum_{i=1}^{\infty} \bar{y}_{2i}(x) \right]^2 - \left[\sum_{i=1}^{m} \bar{y}_{2i}(x) \right]^2 \right\|$$

Physically, $\varepsilon_1^{(n)}$ represents the error of the second-order moment about zero, and $\varepsilon_2^{(n)}$ is the error of the square of the average displacement $\bar{y}^{(n)}(x)$.

The term $\varepsilon_1^{(n)}$ can be written in the form:

$$\varepsilon_1^{(n)} \leqslant 2 \sum_{i=1}^{n} \sum_{j=n+1}^{\infty} \| \langle y_i(\omega, x) y_j(\omega, x)\rangle \| + \sum_{i=n+1}^{\infty} \sum_{j=n+1}^{\infty} \| \langle y_i(\omega, x) y_j(\omega, x)\rangle \|.$$

On the basis of eqns (A2), (A3) and (A4), it is clear that:

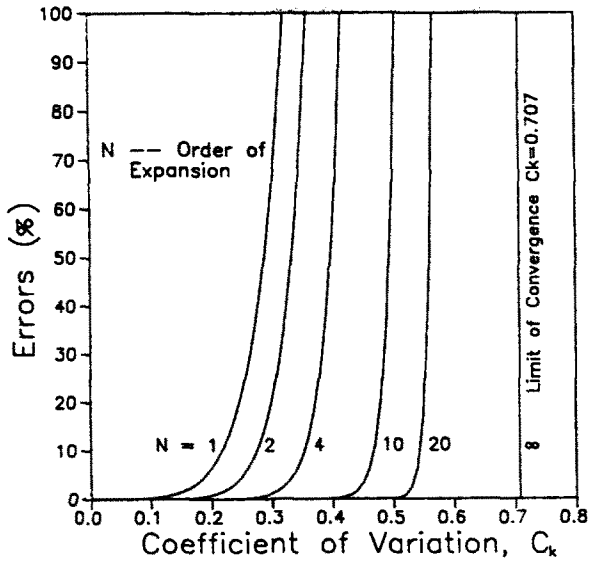

Fig. 6. Upper bound on the errors of the variance function.

$$\| \langle y_i(\omega, x) y_j(\omega, x) \rangle \| \leqslant y_d^2(0)(i+j-1)Z^{j+j}. \tag{48}$$

Using eqn (48) and the technique developed in Appendix A, it is possible to show that:

$$\varepsilon_1^{(n)} \leqslant 2y_d^2(0)\left[\frac{Z}{1-Z}\right]^2\left[2\left(n+\frac{1+Z}{1-Z}\right)-Z^n\left(2n+\frac{1+Z}{1-Z}\right)\right]Z^n. \tag{49}$$

Similarly, recalling that:

$$\| \bar{y}_{2i}(x) \| = \| \langle y_{2i}(\omega, x) \rangle \| \leqslant y_d(0)(2i-1)Z^{2i}$$

it is possible to show:

$$\varepsilon_2^{(n)} \leqslant y_d^2(0)\left[\frac{Z^2}{1-Z^2}\right]^2\left(2m+\frac{1+Z^2}{1-Z^2}\right)\left[2\frac{1+Z^2}{1-Z^2}-Z^{2m}\left(2m+\frac{1+Z^2}{1-Z^2}\right)\right]Z^{2m}. \tag{50}$$

Equations (49), (50) show that both $\varepsilon_1^{(n)}$ and $\varepsilon_2^{(n)}$ approach zero when $n \to \infty$ if $Z < 1$. Therefore the series representation for the variance [eqn (45)] converges provided that $C_k < 1/\sqrt{2}$. In Fig. 6 we show an upper bound to $\varepsilon_V^{(n)}$ as a function of C_k for different n values. This upper bound was calculated by substituting eqns (49) and (50) into eqn (47).

Comparison of Figs 1 and 6 shows that the errors associated with the variance function are much larger than the errors of the average solution. One has to realize however that:

- Figure 6 represents the errors of the variance; the errors of the standard deviation are much smaller, probably of the order of $\sqrt{\varepsilon_V^{(n)}}$.
- One can probably tolerate larger errors in the standard deviation of the solution than in the average solution.

Based on these considerations we take as the acceptable variance error the value of 30%. With this criterion the first-order approximation (which is the same as the small fluctuation solution) is valid if $C_k \leqslant 0.2$. In order to obtain a solution which is valid up to $C_k = 0.5$, a 10th-order approximation is necessary. These estimates are probably conservative due to the conservative nature of the bounding procedure used in the error estimates.

SUMMARY AND CONCLUSION

The classical model of a beam on elastic (Winkler-type) support is quite useful in a variety of geotechnical problems. Due to the natural variability of most natural soils and the inherent limitation in the density of field testing, there exists a significant uncertainty with respect to the true form of the spatial distribution of the coefficient of subgrade reaction k. Such an uncertainty can be incorporated into the analysis by considering this coefficient as a homogeneous random function of the space coordinate along the beam, i.e. $k = k(\omega, x)$. Specifically, the present work deals with the case when $k(\omega, x)$ is a normal function with a "low-pass" spectrum. In this setting the problem is governed by a differential equation with a random parameter on the L.H.S. We analyze this problem using a stochastic functional expansion which has been proposed by Adomian. It is shown that the expansion converges (in the mean square sense) if the coefficient of variation of $k(\omega, x)$, (C_k) is less than $1/\sqrt{2}$. This limit is a sufficient but probably not necessary condition for the convergence of the expansion. For $C_k < 1/\sqrt{2}$ it is possible to show that all finite-order moments of the solution converge.

Explicit expressions and numerical results are presented for the 4th-order approximation of the average solution. We derive upper bounds on the errors associated with various order approximations of the average and variance of the solution. Based on these estimates the 4th-order approximation of the average solution is valid up to approximately $C_k \leqslant 0.5$. All aspects of the solution are governed by two parameters with very clear physical meaning; the coefficient of variation $C_k = \sigma_k/k_0$, and $\Theta = R_k/\lambda$, where σ_k, k_0 and R_k are the standard deviation, average value and autocorrelation distance of $k(\omega, x)$ respectively, and λ is the (deterministic) characteristic length of the beam support system. It is demonstrated that the average solution of the random differential equation tends to the solution of the associated deterministic problem if either C_k or Θ become less than approximately 0.1. The case of small Θ represents the "white noise" limit of the system. Variation of Θ above the value of approximately 10 has no affect on the average solution. Θ values in this high range may be considered to represent the "random variable" limit of the stochastic system. These terms refer to the complete beam–support system, not only to the function $k(\omega, x)$; this is reflected by the fact that the parameter of significance of $\Theta = R_k/\lambda$ rather than only R_k.

A framework for the evaluation of the variance of the solution was derived without the presentation of numerical results. The error analysis shows that the variance errors are much larger than the errors of the average solution.

We have shown that previous solutions of the problem of a beam on random elastic support which are essentially first-order perturbation solutions are represented by the first term in the present expansion. These solutions may be considered valid only if $C_k \leqslant 0.1$–0.15.

We conclude that Adomian's decomposition is both a powerful and convenient tool for the analysis of a certain class of random differential equations, and its application to various other problems should be encouraged.

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APPENDIX A—AN UPPER BOUND ON THE MEAN SQUARE ERROR

We have shown that the solution series [eqn (15)] converges in the mean square sense if the following requirement [eqns (20), 21)] holds:

$$\lim_{n\to 0} A^{(n)} = 0$$

$$A^{(n)} = \langle [e^{(n)}(\omega)]^2 \rangle = \left\| \sum_{i=n+j}^{\infty} \sum_{j=n+1}^{\infty} \langle y_i(\omega, x) y_j(\omega, x) \rangle \right\|.$$

The expression for $A^{(n)}$ implies:

$$A^{(n)} \leqslant \sum_{i=n+1}^{\infty} \sum_{j=n+1}^{\infty} \| \langle y_i(\omega, x) y_j(\omega, x) \rangle \|. \tag{A1}$$

Consider the term $\langle y_i(\omega, x) y_j(\omega, x) \rangle$; using the general term in eqn (18) gives:

$$\langle y_i(\omega, x) y_j(\omega, x) \rangle = \frac{y_d^2(0)(-1)^{i+j}}{(2\lambda k_0)^{i+j}} \int_{u_1=-\infty}^{\infty} \cdots \int_{u_i=-\infty}^{\infty} \int_{u_{i+1}=-\infty}^{\infty} \cdots \int_{u_{i+j}=-\infty}^{\infty} [g_\lambda(x-u_1), \ldots, g_\lambda(u_{i-1}-u_i)] g_\lambda(u_i)$$
$$\times [g_\lambda(x-u_{i+1}), \ldots, g_\lambda(u_{i+j-1}-u_{i+j})] g_\lambda(u_{i+j}) \langle \hat{k}(\omega, u_1), \ldots, \hat{k}(\omega, u_{i+j}) \rangle (\mathrm{d}u_{i+j}, \ldots, \mathrm{d}u_1). \tag{A2}$$

By assumption $\hat{k}(\omega, x)$ is a zero mean, normal function; hence it has the following correlation structure:

$$\langle \hat{k}(\omega, u_1), \ldots, \hat{k}(\omega, u_m) \rangle = \Delta_m \sigma_k^m \sum_{l=1}^{m-1} a_{m,l} \tag{A3}$$

where

$$\Delta_m = \begin{cases} 1 & \text{if } m \text{ is even} \\ 0 & \text{if } m \text{ is odd.} \end{cases} \tag{A4}$$

Each term $a_{m,l}$, $l = 1, \ldots, (m-1)$, is a product of $(m/2)$ autocorrelation functions $\rho_k(u_p - u_q)$ evaluated at all the possible combinations $(u_p - u_q)$ of $(u_1, \ldots, u_m)$.

When $m = 2$ there is only a single term $a_{2,1} = \rho_k(u_1 - u_2)$.

For $m = 4$ we have

$$\langle \hat{k}(\omega, u_1), \ldots, \hat{k}(\omega, u_4) \rangle = \sigma_k^4 [a_{4,1} + a_{4,2} + a_{4,3}]$$

where

$$a_{4,1} = \rho_k(u_1 - u_2)\rho_k(u_3 - u_4)$$
$$a_{4,2} = \rho_k(u_1 - u_3)\rho_k(u_2 - u_4)$$
$$a_{4,3} = \rho_k(u_1 - u_4)\rho_k(u_2 - u_3).$$

Similarly $\langle \hat{k}(\omega, u_1), \ldots, \hat{k}(\omega, u_6) \rangle$ is the sum of five terms, each one of which is the product of three autocorrelation functions.

Substituting (A3) and (A4) into (A2) and the result into (A1) gives:

$$A^{(n)} \leqslant y_d^2(0) \sum_{i=n+1}^{\infty} \sum_{j=n+1}^{\infty} \frac{(C_k)^{i+j} |(-1)^{i+j} \Delta_{i+j}|}{(2\lambda)^{i+j}} \int_{u_1=-\infty}^{\infty} \cdots \int_{u_{i+j}=-\infty}^{\infty} [\| g_\lambda(x-u_1) \|, \ldots, \| g_\lambda(u_{i-1}-u_i) \|] \| g_\lambda(u_i) \|$$
$$\times [\| g_\lambda(x-u_{i+1}) \|, \ldots, \| g_\lambda(u_{i+j-1}-u_{i+j}) \|] \| g_\lambda(u_{i+j}) \| \left[\sum_{l=1}^{i+j-1} \| a_{i+j,l} \| \right] (\mathrm{d}u_{i+j}, \ldots, \mathrm{d}u_1). \tag{A5}$$

The following bounds are used:

$$|(-1)^{i+j}\Delta_{i+j}| = 1 \tag{A6}$$

$$\|g_\lambda(u)\| \leqslant 1 \tag{A7}$$

$$\|\rho_k(u)\| \leqslant 1. \tag{A8}$$

The relation $\|\rho_k(u)\| \leqslant 1$ implies that $\|a_{i+j,l}\| \leqslant 1$. Hence

$$\sum_{l=1}^{i+j-1} \|a_{i+j,l}\| \leqslant (i+j-1). \tag{A9}$$

Substituting eqns (A6)–(A9) into (A5) gives

$$A^{(n)} \leqslant y_d^2(0) \sum_{i=n+1}^{\infty} \sum_{j=n+1}^{\infty} \frac{(C_k)^{i+j}(i+j-1)}{(2\lambda)^{i+j}} \int_{u_1=-\infty}^{\infty} \cdots \int_{u_{i+j}=-\infty}^{\infty} [\|g_\lambda(x-u_1)\|, \ldots, \|g_\lambda(u_{i-1}-u_i)\|] \times [\|g_\lambda(x-u_{i+1})\|, \ldots, \|g_\lambda(u_{i+j-1}-u_{i+j})\|](du_{i+j}, \ldots, du_1). \tag{A10}$$

Recall that $g_\lambda(u) = [\sin(|u/\lambda|)+\cos(|u/\lambda|)]\,e^{-|u/\lambda|}$ [eqn (6)]. The maximum value of $[\sin(|u/\lambda|)+\cos(|u/\lambda|)]$ is $\sqrt{2}$, hence the Green function $g_\lambda(u)$ is bounded by the expression:

$$\|g_\lambda(u)\| \leqslant \sqrt{2}\, e^{-|u/\lambda|}. \tag{A11}$$

It is easily verified that:

$$\int_{u_i=-\infty}^{\infty} \|g_\lambda(u_{i-1}-u_i)\|\, du_i \leqslant \sqrt{2} \int_{u_i=-\infty}^{\infty} e^{-|(u_{i-1}-u_i)/\lambda|}\, du_i = 2\sqrt{2}\lambda.$$

Since this integral does not depend on u_{i-1} it follows that it is possible to evaluate the $(i+j)$ integrals appearing in eqn (A10) one after the other. Each one of these integrals contributes $2\sqrt{2}\lambda$ to the result, so that the $(i+j)$th multiple integral equals merely $(2\sqrt{2}\lambda)^{i+j}$, and eqn. (A10) becomes:

$$A^{(n)} \leqslant y_d^2(0) \sum_{i=n+1}^{\infty} [\sqrt{2}C_k]^i \sum_{j=n+1}^{\infty} (i+j-1)[\sqrt{2}C_k]^j.$$

This equation can be written also as:

$$A^{(n)} \leqslant y_d^2(0) \sum_{i=n+1}^{\infty} Z^i \left[(i-1) \sum_{j=n+1}^{\infty} Z^j + \sum_{j=n+1}^{\infty} jZ^j \right] \tag{A12}$$

where

$$Z = \sqrt{2}C_k. \tag{A13}$$

Evidently series of the form $\sum_{i=n+1}^{\infty} Z^i$ diverge if $Z \geqslant 1$. However, for $Z < 1$ it is possible to establish the following limits:

$$\sum_{i=n+1}^{\infty} Z^i = \frac{Z^{n+1}}{1-Z} \tag{A14}$$

$$\sum_{j=n+1}^{\infty} jZ^j = \left[n + \frac{1}{1-Z} \right] \frac{Z^{n+1}}{1-Z}. \tag{A15}$$

Inserting these sums into (A12), we get finally:

$$A^{(n)} \leqslant y_d^2(0) \left[\frac{Z}{1-Z} \right]^2 \left[2n + \frac{1+Z}{1-Z} \right] Z^{2n}. \tag{A16}$$

We have used the restriction $Z < 1$ in order to establish eqns (A14) and (A15). Equation (A16) shows, however, that the same restriction guarantees that $\lim_{n\to\infty} A^{(n)} = 0$. This limit is exactly the criterion for mean square convergence of the solution series [see eqns (20) and (21)]. We may conclude therefore that $Z < 1$ is a sufficient condition for the validity of Adomian's procedure for the present problem. Using eqn (A13), the convergence requirement is simply:

$$C_k < 1/\sqrt{2}. \tag{A17}$$

APPENDIX B—THE FOURTH-ORDER TERM

Considering the first of eqns (40) and eqn (41) one can see that the term $H_{4,1}(l)$ is the Fourier Transform of a quadruple convolution integral. Hence utilizing the same procedure used for the derivation of $\bar{Y}_2(l)$ we get:

$$H_{4,1}(l) = G_\lambda^3(l)T^2(l)$$

$$= \frac{R_k^2 G_\lambda^3(l)}{\pi^2} \int_{l_1=-\pi\cdot 2R_k}^{\pi\cdot 2R_k} \int_{l_2=-\pi/2R_k}^{\pi/2R_k} G_\lambda(l-l_1)G_\lambda(l-l_2)\,\mathrm{d}l_2\,\mathrm{d}l_1 \tag{B1}$$

The arguments of $H_{4,2}(l)$ and $H_{4,3}(l)$ are not in the form of multiple convolution integrals since the arguments of the autocorrelation functions are in the "wrong" order. In order to overcome this difficulty, consider the variance function of $k(\omega, x)$ as the inverse Fourier Transform of the spectrum; using eqn (3) gives:

$$\rho_k(x) = \frac{R_k}{\pi} \int_{l=-\pi/2R_k}^{\pi/2R_k} \mathrm{e}^{\mathrm{i}lx}\,\mathrm{d}l$$

where $\mathrm{i} = \sqrt{-1}$; therefore:

$$\rho_k(u_1-u_3) = \frac{R_k}{\pi} \int_{l=-\pi/2R_k}^{\pi/2R_k} \mathrm{e}^{\mathrm{i}l(u_1-u_3)}\,\mathrm{d}l$$

$$= \frac{R_k}{\pi} \int_{l=-\pi/2R_k}^{\pi/2R_k} \mathrm{e}^{\mathrm{i}l(u_1-u_2)}\,\mathrm{e}^{\mathrm{i}l(u_2-u_3)}\,\mathrm{d}l$$

and

$$\rho_k(u_2-u_4) = \frac{R_k}{\pi} \int_{l=-\pi/2R_k}^{\pi/2R_k} \mathrm{e}^{\mathrm{i}l(u_2-u_3)}\,\mathrm{e}^{\mathrm{i}l(u_3-u_4)}\,\mathrm{d}l.$$

Substituting these relations into eqn (40),

$$H_{4,2}(l) = \left[\frac{R_k}{\pi}\right]^2 \int_{l_1=-\pi/2R_k}^{\pi/2R_k} \int_{l_2=-\pi/2R_k}^{\pi/2R_k} \mathbf{F}_T\left\{\int_{u_1=-\infty}^{\infty} \cdots \int_{u_4=-\infty}^{\infty} [g_\lambda(x-u_1)][g_\lambda(u_1-u_2)\,\mathrm{e}^{\mathrm{i}l_1(u_1-u_2)}]\right.$$

$$\left.\times [g_\lambda(u_2-u_3)\,\mathrm{e}^{\mathrm{i}l_1(u_2-u_3)}\,\mathrm{e}^{\mathrm{i}l_2(u_2-u_3)}][g_\lambda(u_3-u_4)\,\mathrm{e}^{\mathrm{i}l_2(u_3-u_4)}]\,(\mathrm{d}u_4,\ldots,\mathrm{d}u_1)\right\}\mathrm{d}l_2\,\mathrm{d}l_1.$$

The operand of the Fourier Transform in this equation is now in the form of a multiple convolution integral, and we get:

$$H_{4,2}(l) = \left[\frac{R_k}{\pi}\right]^2 G_\lambda^2(l) \int_{l_1=-\pi/2R_k}^{\pi/2R_k} \int_{l_2=-\pi/2R_k}^{\pi\cdot 2R_k} G_\lambda(l-l_1)G_\lambda(l-l_2)G_\lambda(l-l_1-l_2)\,\mathrm{d}l_2\,\mathrm{d}l_1. \tag{B2}$$

During the derivation of eqn. (B2) use was made of the "shifting theorem" for Fourier Transform in the form:

$$\mathbf{F}_T\{g_\lambda(x)\,\mathrm{e}^{\mathrm{i}l_1x}\} = G_\lambda(l-l_1).$$

Using a similar procedure for the evaluation of $H_{4,3}$ gives:

$$H_{4,3}(l) = \left[\frac{R_k}{\pi}\right]^2 G_\lambda^3(l) \int_{l_1=-\pi\cdot 2R_k}^{\pi\cdot 2R_k} \int_{l_2=-\pi/2R_k}^{\pi/2R_k} G_\lambda(l-l_1)G_\lambda(l-l_1-l_2)\,\mathrm{d}l_2\,\mathrm{d}l_1. \tag{B3}$$